ON PERTURBATION THEORY OF ELECTROMAGNETIC CAVITY RESONATORS

by

C. H. Papas

Published in
J. App. Phys. 25 (1552-1553) 1954.

Antenna Laboratory

Technical Report No. 3

CALIFORNIA INSTITUTE OF TECHNOLOGY

Pasadena, California

A Technical Report to the Office of Naval Research

Prepared for

Office of Naval Research Contract

Nonr 220(14)

NR 071-262

March 1954

ON PERTURBATION THEORY OF ELECTROMAGNETIC CAVITY RESONATORS

Abstract

In this note the Lagrangian function for the electromagnetic field of a cavity resonator is found. And from this Lagrangian is deduced a perturbation formula which includes Müller's celebrated result as a special case. The same perturbation formula is derived also from the Boltzmann-Ehrenfest adiabatic theorem in a most simple manner.

Introduction

If a cavity resonator has a "simple" shape and is filled with a homogeneous, isotropic medium, to calculate its resonant frequencies and mode functions is a straightforward task in principle. However, it is sometimes of interest to determine the oscillatory properties of a cavity that differs by a small amount in one or more of its physical characteristics from a cavity the oscillatory properties of which are known. For example, the calculation of resonant frequency shift accompanying a slight deformation of the cavity's walls or an introduction of a small foreign object such as a ferrite or semiconductor in the cavity's volume is necessary in certain measurement techniques.

We accordingly derive a formula which allows the calculation of the shift in resonant frequency produced by a foreign body of given size, shape, dielectric constant, and permeability. We start from the Lagrangian of the electromagnetic field and deduce a general perturbation relation, a special application of which leads to the desired perturbation formula.[1]

The adiabatic limitations of the perturbation formula are discussed in the sense of the Boltzmann-Ehrenfest adiabatic invariant theorem. And it is shown that within the framework of these limitations the only calculations that have to be made are those of determining the magnetization and polarization of a body immersed in a uniform static field.

FIELDS IN CAVITY RESONATORS OF SIMPLE SHAPE

A cavity resonator is said to have a "simple" shape when each of its surfaces is a coordinate surface in a space which allows separation of the wave equation. For these simple shapes it is possible to determine the resonant frequencies and mode functions explicitly. Since some general properties of the cavity fields will be essential to the development of our deductions, we shall just briefly review them here.

The electromagnetic field in a lossless cavity (of any shape) forms a standing wave. No traveling waves can exist because no energy is fed into or extracted from the cavity. Consequently, the electric field is everywhere of the same phase. Accordingly, we write the electric field as

$$\underline{E}_o = \underline{E}\, e^{-i\omega t} \quad , \tag{1}$$

and in consequence of this we can write the magnetic field as

$$\underline{H}_o = -i\, \underline{H}\, e^{-i\omega t} \, , \tag{2}$$

where $\underline{E}$ and $\underline{H}$ are real vectors. Substituting (1) and (2) into the two curl equations of Maxwell, we see that $\underline{E}$ and $\underline{H}$ must satisfy the following real, i.e., not complex, equations:

$$\nabla \times \underline{E} = \omega\mu \underline{H} \; , \tag{3}$$

$$\nabla \times \underline{H} = \omega\varepsilon \underline{E} \; , \tag{4}$$

where ε and μ are the dielectric constant and permeability of the homogeneous, isotropic medium filling the cavity.

It follows from (1) and (2) that the time-average electric and magnetic energy densities are respectively given by

$$\overline{U}_e = \frac{1}{4} \; \underline{E} \cdot \underline{E} \tag{5}$$

and

$$\overline{U}_m = \frac{1}{4} \mu \, \underline{H} \cdot \underline{H} \; . \tag{6}$$

The total electric and magnetic energies are integrals of $\overline{U}_e$ and $\overline{U}_m$ throughout the volume of the cavity and are respectively denoted by $\overline{W}_e$ and $\overline{W}_m$. As is well-known, at resonance

$$\overline{W}_e = \overline{W}_m \; . \tag{7}$$

Lagrangian of the Electromagnetic Field

Let us consider the following expression

$$L = \int_V \left[i\omega\mu \frac{1}{2} \; \underline{H}^2 - i\omega\varepsilon \; \frac{1}{2} \underline{E}^2 - \underline{E} \cdot \nabla \times \underline{H} + \frac{1}{2} \sigma \underline{E} + \underline{J}^e \cdot \underline{E} - \underline{J}^m \cdot \underline{H} \right] dV \tag{8}$$

where σ, μ, ε are respectively the conductivity, permeability, and dielectric constant of the medium filling the volume V. $\underline{J}^e$ is the electric current density and $\underline{J}^m$ is the magnetic current density. And $\underline{E}$ and $\underline{H}$ denote the electric and magnetic fields respectively. The harmonic time dependence $e^{-i\omega t}$ has been suppressed.

We recall that the two curl equations of Maxwell are

$$\nabla \times \underline{E} = i\omega\mu \, \underline{H} - \underline{J}^m \; , \tag{9}$$

$$\nabla \times \underline{H} = -i\omega\varepsilon \, \underline{E} + \sigma \underline{E} + \underline{J}^e \; . \tag{10}$$

We now compute δL, the first variation of L, produced by the first variations, $\delta\underline{E}$ and $\delta\underline{H}$, of the electromagnetic field. Taking the first variation of (8) we get

$$\delta L = \int_V \left[i\omega\mu \underline{H} \cdot \delta\underline{H} - i\omega\varepsilon \underline{E} \cdot \delta\underline{E} - \delta\underline{E} \cdot \nabla \times \underline{H} - \underline{E} \cdot \nabla \times \delta\underline{H} + \sigma \underline{E} \cdot \delta\underline{E} + \underline{J}^e \cdot \delta\underline{E} - \underline{J}^m \cdot \delta\underline{H} \right] dV. \tag{11}$$

With the aid of the vector identity,

$$-\underline{E} \cdot \nabla \times \delta\underline{H} = \nabla \cdot (\underline{E} \times \delta\underline{H}) - (\nabla \times \underline{E}) \cdot \delta\underline{H} \quad ,$$

we rewrite (11) in the following form:

$$\delta L = \int_V \left[\delta\underline{H} \cdot (i\omega\mu \underline{H} - \nabla \times \underline{E} - \underline{J}^m) + \delta\underline{E} \cdot (-i\omega\varepsilon \underline{E} - \nabla \times \underline{H} + \sigma \underline{E} + \underline{J}^e) \right] dV + \int_V \nabla \cdot (\underline{E} \times \delta\underline{H}) \, dV \quad . \tag{11 a}$$

We see that the coefficients of $\delta\underline{H}$ and $\delta\underline{E}$ are zero by virtue of the Maxwell equations (9) and (10), and hence,

$$\delta L = \int_V \nabla \cdot (\underline{E} \times \delta\underline{H}) \, dV \quad . \tag{12}$$

If $\underline{n}$ is the outward normal of the surface S bounding the volume V, by the divergence theorem (12) becomes

$$\delta L = \int_S \underline{n} \cdot (\underline{E} \times \delta\underline{H}) \, dS \ . \tag{13}$$

If S is a perfectly conducting surface, then $\underline{n} \times \underline{E}$ vanishes on S and $\delta L = 0$. Or if $\delta\underline{H}$ is restricted so that $\underline{n} \times \delta\underline{H} = 0$ on S, again $\delta L = 0$. If at least one of these conditions is satisfied, then (8) is the Lagrangian of the field and (9) and (10) are its Euler-Lagrange equations.

Perturbation Relation

Suppose that volume V is filled with a medium of constitutive parameters μ_a, ε_a, σ_a. And within this medium we have sources $\underline{J}_a^e$ and

$\underline{J}_a^m$ oscillating at frequency ω_a . Then the corresponding field quantities are $\underline{E}_a$ and $\underline{H}_a$. And further suppose that if the volume were filled with medium μ_b, ε_b, σ_b, the sources $\underline{J}_b^e$ and $\underline{J}_b^m$ oscillating at frequency ω_b would give rise to the fields $\underline{E}_b$ and $\underline{H}_b$. In other words, we are considering the volume V under two different states, "a" and "b" . First we consider the volume V to be in state a, and we restrict the variations of $\underline{E}_a$ and $\underline{H}_a$ such that

$$\delta\, \underline{E}_a = -\lambda \underline{E}_b^* \tag{14}$$

$$\delta\, \underline{H}_a = \lambda\, \underline{H}_b^* \quad . \tag{15}$$

In other words we restrict the variations of the electric and magnetic fields of state a to be proportional to the conjugate complex of the fields in state b. λ is a proportionality constant. Substituting (14) and (15) into (11) we get

$$\delta L = \lambda \int_V \Big[i\,\omega_a\, \mu_a\, \underline{H}_a \cdot \underline{H}_b^* + i\,\omega_a\, \varepsilon_a\, \underline{E}_a \cdot \underline{E}_b^* + \underline{E}_b^* \cdot \nabla \times \underline{H}_a - \underline{E}_a \cdot \nabla \times \underline{H}_b^* -$$

$$- \sigma_a\, \underline{E}_a \cdot \underline{E}_b^* - \underline{J}_a^e \cdot \underline{E}_b^* - \underline{J}_a^m \cdot \underline{H}_b^* \Big]\, dV \quad . \tag{16}$$

Now consider the volume V to be in state b and restrict the variations of $\underline{E}_b$ and $\underline{H}_b$ such that $\delta\underline{E}_b$ and $\delta\underline{H}_b$ are proportional to the conjugate complex of the fields in state a :

$$\delta\underline{E}_b = -\lambda\, \underline{E}_a^* \tag{17}$$

$$\delta\underline{H}_b = \lambda\, \underline{H}_a^* \tag{18}$$

Substituting (17) and (18) in (11) we get

$$\delta L = \lambda \int_V \Big[i\omega_b \mu_b \underline{H}_b \cdot \underline{H}_a^* + i\omega_b \epsilon_b \underline{E}_b \cdot \underline{E}_a^* + \underline{E}_a^* \cdot \nabla \times \underline{H}_b - \underline{E}_b \cdot \nabla \times \underline{H}_a^* - \sigma_b \underline{E}_b \cdot \underline{E}_a^* - \underline{J}_b^e \cdot \underline{E}_a^* - \underline{J}_b^m \cdot \underline{H}_a^* \Big] dV. \tag{19}$$

We know that the left side of (16) can be written as

$$\delta L = \lambda \int_V \nabla \cdot (\underline{E}_a \times \underline{H}_b^*)\, dV \tag{20}$$

when (15) is substituted into (13); and similarly the left side of (19) can be written as

$$\delta L = \lambda \int_V \nabla \cdot (\underline{E}_b \times \underline{H}_a^*)\, dV \tag{21}$$

when (18) is substituted into (13). Consequently, if we add the conjugate complex of (19) to (16), we get

$$\int_V \nabla \cdot (\underline{E}_a \times \underline{H}_b^* + \underline{E}_b^* \times \underline{H}_a)\, dV = \int_V \Big[i(\omega_a \mu_a - \omega_b \mu_b)\, \underline{H}_a \cdot \underline{H}_b^* + i(\omega_a \varepsilon_a - \omega_b \varepsilon_b)\, \underline{E}_a \cdot \underline{E}_b^* - (\sigma_a + \sigma_b)\, \underline{E}_a \cdot \underline{E}_b - (\underline{J}_a^e \cdot \underline{E}_b^* + \underline{J}_b^{e*} \cdot \underline{E}_a) - (\underline{J}_a^m \cdot \underline{H}_b^* + \underline{J}_b^{m*} \cdot \underline{H}_a) \Big] dV. \tag{22}$$

This is a most general perturbation relation. We shall use in this note only one of its many applications.

Application of Perturbation Formula to Cavity Resonators

We now apply (22) to cavity resonators. Suppose the volume V is bounded by a perfectly conducting surface S and within V the dielectric constant is ε_a, the permeability is μ_a, and the conductivity σ_a is zero. The cavity is oscillating at a frequency ω_a in a certain mode. Upon introducing a lossless foreign body of small volume τ, the resonant frequency of the cavity changes. We are interested in computing the new resonant frequency.

Since the wall of the cavity is lossless the left side of (22) disappears. This can be seen by transforming the volume integral into an integral over the surface S and noting that $\underline{n} \times \underline{E} = 0$ on perfectly conducting surface. And since the cavity is assumed to be free of sources, all the current densities in (22) are zero. Moreover, the conduct
(22) are zero. Consequently, (22) reduces to

$$0 = \int_V \left[(\omega_a \mu_a - \omega_b \mu_b) \; \underline{H}_a \cdot \underline{H}_b^* + (\omega_a \varepsilon_a - \omega_b \varepsilon_b) \, \underline{E}_a \cdot \underline{E}_b^* \right] dV. \tag{23}$$

$\underline{H}_a$, $\underline{E}_a$, ω_a are the fields and the resonant frequency respectively before insertion of the foreign body, and μ_a, ε_a are the constitutive parameters. After foreign object of constitutive parameters μ_b, ε_b is introduced, the fields become $\underline{E}_b$, $\underline{H}_b$ and the frequency shifts from ω_a to ω_b .

We write

$$\begin{aligned} \omega_b &= \omega_a + \delta\omega_a \;, \qquad && \varepsilon_b = \varepsilon_a + \delta\varepsilon_a \;, \; \mu_b = \mu_a + \delta\mu_a \\ \underline{E}_b &= \underline{E}_a + \delta\underline{E}_a \;, \qquad && \underline{H}_b = \underline{H}_a + \delta\underline{H}_a \end{aligned} \tag{24}$$

where $\delta\omega_a$ is the shift in resonant frequency, $\delta\varepsilon_a$ is the difference between the dielectric constant ε_a of the medium and the dielectric constant ε_b of the foreign body, $\delta\mu_a$ is the difference between the permeability μ_a of the medium surrounding the foreign body and the permeability μ_b of the foreign body. And similarly $\delta\underline{E}_a$ and $\delta\underline{H}_a$ are the perturbations in the field due to the foreign body.

Substituting (24) into (23) we get

$$0 = \int_V (\omega_a \mu_a - \omega_a \mu_a - \mu_a \delta\omega_a - \omega_a \delta\mu_a + \delta\omega_a \delta\mu_a)(\underline{H}_a \cdot \underline{H}_a^* + \underline{H}_a \cdot \delta\underline{H}_a^*)$$

$$+ (\omega_a \varepsilon_a - \omega_a \varepsilon_a - \varepsilon_a \delta\omega_a - \omega_a \delta\varepsilon_a - \delta\omega_a \delta\varepsilon_a)(\underline{E}_a \cdot \underline{E}_a^* + \underline{E}_a \cdot \delta\underline{E}_a^*)\, dV \,. \qquad (25)$$

We neglect quantities of the second order, i.e., $\delta\omega_a \delta\mu_a$ and $\delta\omega_a \delta\varepsilon_a$. Hence, (25) reduces to

$$0 = -\omega_a \int_V \left[\delta\mu_a \underline{H}_a \cdot (\underline{H}_a^* + \delta\underline{H}_a^*) + \delta\varepsilon_a \underline{E}_a \cdot (\underline{E}_a^* + \delta\underline{E}_a^*)\right] dV$$

$$-\delta\omega_a \int_V \left[\mu_a \underline{H}_a \cdot (\underline{H}_a^* + \delta\underline{H}_a^*) + \varepsilon_a \underline{E}_a \cdot (\underline{E}_a^* + \delta\underline{E}_a^*)\right] dV \,, \qquad (26)$$

In the first integral the integration extends throughout the volume of the foreign body only because $\delta\mu_a$ and $\delta\varepsilon_a$ are identically zero outside the foreign body. $\underline{H}_a^* + \delta\underline{H}_a^*$ is the resultant magnetic field inside the foreign body and $\underline{H}_a$ is the undisturbed magnetic field. Similarly, $\underline{E}_a^* + \delta\underline{E}_a^*$ is the electric field within the foreign body and $\underline{E}_a$ is the undisturbed electric field. In the second integral we can neglect $\delta\underline{H}_a^*$ and $\delta\underline{E}_a^*$ because the volume τ of the foreign body, which is the only region where the perturbation fields can possibly have an appreciable value, is small compared to V. Thus we can rewrite (26) as

$$0 = -\omega_a \int_\tau \delta\mu_a \underline{H}_a \cdot (\underline{H}_a^* + \delta\underline{H}_a^*) + \delta \;_a \underline{E}_a \cdot (\underline{E}_a^* + \delta\underline{E}_a^*) \quad dV$$

$$- \delta\omega_a \int_V \left[\mu_a \underline{H}_a \cdot \underline{H}_a^* + \varepsilon_a \underline{E}_a \cdot \underline{E}_a^*\right] dV \,. \qquad (27)$$

If we consider the case where the medium in the cavity is a vacuum and the constitutive parameters of the foreign body are μ_1 and ε_1, then

$$\mu_a = \mu_o , \quad \varepsilon_a = \varepsilon_o , \quad \underline{H}_a = \underline{H}_o , \quad \underline{E}_a = \underline{E}_o ,$$
$$\delta\mu_a = \mu_1 - \mu_o , \quad \delta\varepsilon_a = \varepsilon_1 - \varepsilon_o , \quad \omega_a = \omega_o \tag{28}$$

where ε_o and μ_o are the constitutive parameters of vacuum, $\underline{H}_o$ and $\underline{E}_o$ are the fields in the hollow cavity with no foreign body present. Substituting (28) into (27), we get

$$0 = -\omega_o \int_\tau \left[(\mu_1 - \mu_o)\, \underline{H}_o \cdot (\underline{H}_o^* + \delta\underline{H}_o) + (\varepsilon_1 - \varepsilon_o)\, \underline{E}_o \cdot (\underline{E}_o^* + \delta\underline{E}_o^*)\right] dV$$
$$- \delta\omega_o \int_V (\mu_o\, \underline{H}_o \cdot \underline{H}_o^* + \varepsilon_o\, \underline{E}_o \cdot \underline{E}_o^*)\, dV . \tag{29}$$

We can interpret now the integrand of the first integral of (29) in terms of the polarization $\underline{P}$ and the magnetization $\underline{M}$. We recall that $\underline{D} = \varepsilon_o \underline{E} + \underline{P} = \varepsilon \underline{E}$ or $(\varepsilon - \varepsilon_o)\, \underline{E} = \underline{P}$ and that $\underline{B} = \mu_o \underline{H} + \underline{M} = \mu \underline{H}$ or $(\mu - \mu_o)\, \underline{H} = \underline{M}$. In view of this

$$(\varepsilon_1 - \varepsilon_o)(\underline{E}_o + \delta\underline{E}_o) = \underline{P}$$
$$(\mu_1 - \mu_o)(\underline{H}_o + \delta\underline{H}_o) = \underline{M} . \tag{30}$$

Consequently by substituting (30) into (29) we get

$$- \frac{\delta\omega_o}{\omega_o} = \frac{\int_\tau \left[\underline{H}_o \cdot \underline{M}^* + \underline{E}_o \cdot \underline{P}^*\right] dV}{\int_V \left[\mu_o \underline{H}_o \cdot \underline{H}_o^* + \varepsilon_o \underline{E}_o \cdot \underline{E}_o^*\right] dV} . \tag{31}$$

If we define the total magnetization of the foreign body by $\underline{m}$ and its total polarization by $\underline{p}$, i.e.,

$$\underline{m} = \int_\tau \underline{M}\, dV \tag{32}$$

$$\underline{p} = \int_\tau \underline{P}\, dV , \tag{33}$$

and if we note that the undisturbed fields, $\underline{E}_o$ and $\underline{H}_o$, are essentially homogeneous throughout the small volume τ, (31) becomes

$$- \frac{\delta\omega_o}{\omega_o} = \frac{\underline{m}^* \cdot \underline{H}_o + \underline{p}^* \cdot \underline{E}_o}{4\,\overline{W}_t} \tag{34}$$

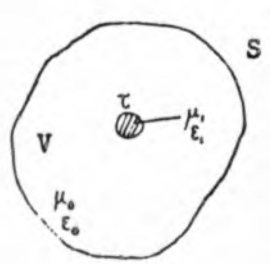
where

$$4\,\overline{W}_t = \int_V (\mu_o\,\underline{H}_o \cdot \underline{H}_o^* + \varepsilon_o\,\underline{E}_o \cdot \underline{E}_o^*)\,dV \quad . \tag{35}$$

$\overline{W}_t$ is the time-average total energy stored in the cavity before the foreign body is introduced.

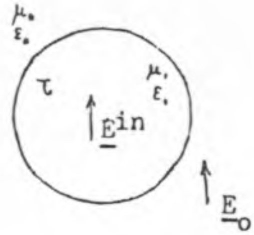
The relation (34) is very useful and easy to apply. Of course, the cavity must be of simple shape so that we can determine explicitly the undisturbed fields $\underline{H}_o$ and $\underline{E}_o$ corresponding to a certain mode of frequency ω_o as well as the time-average total energy $\overline{W}_t$. Since the foreign body is small we can assume that its $\underline{m}$ and $\underline{p}$ are produced respectively by a magnetostatic field and an electrostatic field.

As an example of the application of (34) let us consider a small sphere of dielectric constant ε_1 and permeability μ_1 placed in a cavity resonator otherwise filled with a vacuum.



Let us first consider the electric effect, i.e., let us compute $\underline{p}$. Since the sphere is small $(\tau << V)$ we can consider the problem as being equal to the one of finding $\underline{p}$ for a dielectric sphere in a uniform electrostatic field $\underline{E}_o$. From elementary considerations we know that the resultant electric field within the sphere is

$$\underline{E}^{in} = \frac{3\,\varepsilon_o}{\varepsilon_1 + 2\varepsilon_o}\,\underline{E}_o$$



and
$$\underline{D}^{in} = \varepsilon_1\,\underline{E}^{in} .$$

Since $\underline{P} = \underline{D} - \varepsilon_o\,\underline{E}$, we have

$$\underline{P} = \underline{D}^{in} - \varepsilon_o \underline{E}^{in} = (\varepsilon_1 - \varepsilon_o)\,\underline{E}^{in} = (\varepsilon_1 - \varepsilon_o)\,\frac{3\,\varepsilon_o}{\varepsilon_1 + 2\varepsilon_o}\,\underline{E}_o .$$

Integrating $\underline{P}$ throughout τ we obtain

$$\underline{p} = 3\,\varepsilon_o\,\frac{\varepsilon_1 - \varepsilon_o}{\varepsilon_1 + 2\varepsilon_o}\,\underline{E}_o\,(\tfrac{4}{3}\,\pi a^3) \tag{36}$$

And proceeding in a similar way for a sphere of permeability μ_1 in a magnetostatic field $\underline{H}_o$ we get

$$\underline{m} = 3\,\mu_o\,\frac{\mu_1 - \mu_o}{\mu_1 + 2\mu_o}\,\underline{H}_o\,(\tfrac{4}{3}\,\pi a^3) \tag{37}$$

Substituting (36) and (37) into (34) we get

$$\frac{-\delta\omega_o}{\omega_o} = \frac{3\,\mu_o\,\frac{\mu_1 - \mu_o}{\mu_1 + 2\mu_o}\,\underline{H}_o \cdot \underline{H}_o^* + 3\,\varepsilon_o\,\frac{\varepsilon_1 - \varepsilon_o}{\varepsilon_1 + 2\varepsilon_o}\,\underline{E}_o \cdot \underline{E}_o^*}{4\,\overline{W}_t}\,\Delta V , \tag{38}$$

where $\Delta V = \frac{4}{3}\pi a^3$.

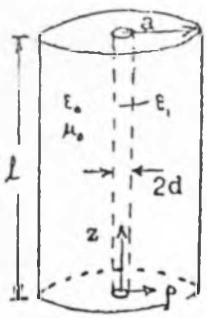
The case of a metallic sphere is a limiting case of (38). Since a perfectly conducting metal can be considered as having $\mu = 0$ and $\varepsilon = \infty$, we need only put $\mu_1 = 0$ and $\varepsilon_1 = \infty$ in (38) to obtain the frequency shift produced by a perfectly conducting sphere:

$$-\frac{\delta\omega_o}{\omega_o} = \frac{-\frac{3}{2}\,\mu_o\,\underline{H}_o \cdot \underline{H}_o^* + 3\,\varepsilon_o\,\underline{E}_o \cdot \underline{E}_o^*}{4\,\overline{W}_t}\,\Delta V . \tag{39}$$

We can qualitatively apply (39) to the problem of frequency shifts produced by inward and outward dents in the cavity's wall. Suppose at a point on the wall, where the magnetic field is zero we produce a small inward dent, say of hemispherical shape. Since the magnetic field was originally zero in the region occupied by the hemispherical dent, we set $\underline{H}_0$ in (39) equal to zero. But then the right side of (39) is a positive quantity. This leads us to conclude that the frequency change is negative, i.e., the resonant frequency of the cavity decreases. If at this same point (where the original magnetic field is zero) we had made an outward dent rather than an inward one, the resonant frequency would have increased. This is easy to see if we consider the cavity with an outward dent as the original cavity and the one with no dent as the perturbed cavity and note that we can pass from the original cavity to the perturbed one by an inward dent. In case a dent is located where the electric field is zero, $\underline{E}_0$ in (39) is equal to zero and the results are just the converse of what they were before: where there is no electric field, an inward indentation increases the frequency and an outward one decreases it.

The application of (34) to small objects of arbitrary shape amounts to a computation of the total magnetization and total polarization of the objects on a static basis. For oblate and prolate spheroids, i.e., for discs and needles, these computations have been performed and can be found in the literature.[2]

As another illustrative example[3] let us consider a cylindrical cavity of radius a and length ℓ oscillating in the TM_{010} mode with a thin dielectric rod of radius d along its axis. The question is, what is the shift in resonant frequency produced by the presence of the rod? When no rod is present we know that the field within the cavity is



$$E_z = J_0(k\rho) \tag{40}$$

$$\sqrt{\frac{\mu_0}{\varepsilon_0}}\; H_\phi = -i\; J_1(k\rho) \tag{41}$$

where k is such that $J_0(ka) = 0$. The first root of the Bessel function is 2.405 and, hence, $k = 2.405/a$ or $\lambda = 2\pi/k = 2.61a$. In the neighborhood of the axis, i.e., $\rho \sim 0$, we see from (41) that H_ϕ is small. Indeed it vanishes at $\rho = 0$. On the other hand E_z is maximum at $\rho = 0$. It is, therefore, reasonable to assume that the electric effect will be the dominant one. We accordingly neglect $\underline{m}^* \cdot \underline{H}_0$ in (34). Thus

$$-\frac{\delta\omega_0}{\omega_0} = \frac{\underline{p}^* \cdot \underline{E}_0}{2\varepsilon_0 \int_V \underline{E}_0 \cdot \underline{E}_0^* \, dV} \tag{42}$$

where we have used the fact that $\int_V \mu_0 \underline{H}_0 \cdot \underline{H}_0^* \, dV = \int_V \varepsilon_0 \underline{E}_0 \cdot \underline{E}_0^* \, dV$.

But from the first of equations (30) and from equation (33), we know that

$$\underline{p} = \int_V (\varepsilon_1 - \varepsilon_0)(\underline{E}_0 + \delta\underline{E}_0)\, dV. \tag{43}$$

Upon neglecting $\delta\underline{E}_0$ in (43), we get

$$\underline{p} = \int_V (\varepsilon_1 - \varepsilon_0)\, \underline{E}_0 \, dV . \tag{44}$$

And it follows from (42) and (44) that

$$-\frac{\delta\omega_0}{\omega_0} = \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_0} - 1\right) \frac{\int_{rod} \underline{E}_0 \cdot \underline{E}_0^* \, dV}{\int_{cavity} \underline{E}_0 \cdot \underline{E}_0^* \, dV} \tag{45}$$

ε_1 is the dielectric constant of the rod and may be complex if its conductivity is different from zero. According to (40) $\underline{E}_o \cdot \underline{E}_o^* = J_o^2(k\rho)$. Subsitituting this into (45) we find that

$$- \frac{\delta\omega_o}{\omega_o} = \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_o} - 1\right) \frac{\int_0^{\ell}\int_0^{d} J_o^2(k\rho)\; 2\pi\rho\, d\rho\, dz}{\int_0^{\ell}\int_0^{a} J_o^2(k\rho)\; 2\pi\rho\, d\rho\, dz} \quad . \tag{46}$$

Carrying out the straightforward integrations, we finally obtain an explicit formula for the shift in resonant frequency produced by the dielectric rod:

$$- \frac{\delta\omega_o}{\omega_o} = \frac{1}{2J_1^2(ka)} \left(\frac{d}{a}\right)^2 \left(\frac{\varepsilon_1}{\varepsilon_o} - 1\right) \tag{47}$$

Adiabatic Considerations

In the foregoing deductions and in the illustrative examples we have assumed tacitly that the change of the resonant field is "adiabatic". By this we mean that the number of nodal surfaces in the field pattern is not changed by the foreign body. If the volume occupied by the foreign body is small compared to spatial variations of the field, the undisturbed field is homogeneous throughout the volume and the disturbing influence of the foreign body is not sufficiently strong to force a jump from the original mode to an adjacent one. Therefore, the restriction that the foreign body be small is equivalent to limiting the allowable changes to adiabatic ones.

It may appear at first that the problem of a dielectric rod in a cylindrical cavity violates this restriction. However, this is not the case because the TM_{010} mode does not vary along the rod; it varies only across the rod and in this dimension the rod is indeed assumed to be small. If, on the other hand, we were to increase the conductivity of the rod, we would find that eventually the rod would be more like a conductor than a dielectric, and consequently the original TM_{010} mode would jump to a coaxial line mode.

Application of the Adiabatic Theorem

According to the Boltzmann-Ehrenfest[4] adiabatic theorem, if the state of any oscillating system is changed adiabatically, the product of the period and the time average energy remains invariant. That is,

$$\frac{\overline{W}_t}{\omega} = C = \text{invariant} \qquad (48)$$

where C is a constant. It follows from (48) that

$$\frac{\delta \overline{W}_t}{\overline{W}_t} = \frac{\delta\omega}{\omega} \qquad (49)$$

If we note that $\delta\overline{W}_t$ is the change in time average energy produced by the foreign body, then

$$\delta\overline{W}_t = -\frac{1}{4}\,\underline{m}^* \cdot \underline{H}_o - \frac{1}{4}\,\underline{p}^* \cdot \underline{E}_o \qquad (50)$$

because $\frac{1}{4}\,\underline{m}^* \cdot \underline{H}_o$ and $\frac{1}{4}\,\underline{p}^* \cdot \underline{E}_o$ are the energy changes produced by a small body placed respectively in a magnetostatic field equal to $\underline{H}_o$ and in an electrostatic field equal to $\underline{E}_o$. And finally from (49) and (50) we obtain

$$-\frac{\delta\omega}{\omega} = \frac{\underline{m}^* \cdot \underline{H}_o + \underline{p}^* \cdot \underline{E}_o}{4\overline{W}_t} \qquad (51)$$

It is to be noted that (51) is identical to (34).

References:

1. As was so kindly pointed out by Professor F. Borgnis, this perturbation formula was originally derived in a different manner by Dr. Johannes Müller. The reader is referred to Müller's interesting paper "Untersuchung uber elektromagnetische Hohlraume", Hochfrequenztechnik and Elektroakustik, 54, 157-161 (1939), and to a readable paraphrase by H. B. G. Casimir, "Theory of Electromagnetic Waves in Resonant Cavities" Philips Research Report, 6, 162-182 (1951).

2. S. A. Schelkunoff and H. T. Friis, "Antennas", p. 580-581, Wiley, New York, 1952.

   W. R. Smythe, "Static and Dynamic Electricity", McGraw-Hill, New York, 1951.

3. F. Borgnis, Hochfrequenztechnik and Elektroakustic, 59 (1942) 22-26.

4. See page 183-185, in "Les Tenseurs" by L. Brillouin, Dover, New York 1946.